3.3. PARTIAL DERIVATIVES OF FIRST ORDER

Let $z = f(x, y)$ be a function of two independent variables x and y. If y is kept constant and x alone is allowed to vary, then z becomes a function of x only. The derivative of z, with respect to x, treating y as constant, is called partial derivative of z w.r.t. x and is denoted by

$$\frac{\partial z}{\partial x} \quad \text{or} \quad \frac{\partial f}{\partial x} \quad \text{or} \quad f_x.$$

Thus, $$\frac{\partial z}{\partial x} = \underset{h \to 0}{\text{Lt}} \frac{f(x+h, y) - f(x, y)}{h}$$

Similarly, the derivative of z, with respect to y, treating x as constant, is called partial derivative of z w.r.t. y and is denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or f_y.

Thus, $$\frac{\partial z}{\partial y} = \underset{k \to 0}{\text{Lt}} \frac{f(x, y+k) - f(x, y)}{k}$$

$\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are called **first order partial derivatives of z.**

[**In general,** if z is a function of two or more independent variables, then the **partial derivative of z** w.r.t. **any one** of the independent variables is the **ordinary derivative of z** w.r.t. that variable, **treating all other variables as constant.**]

Geometrically. Let $z = f(x, y)$ be a function of two variables x and y. Then by Art. 3.1, it represents a surface S. If $y = k$, a constant, then $y = k$ represents a plane parallel to the zx-plane.

$\therefore$ $z = f(x, y)$ and $y = k$ represent a plane curve C which is the section of S by $y = k$.

$\frac{\partial z}{\partial x}$ represents the slope of tangent to C at (x, k, z).

Thus, $\frac{\partial z}{\partial x}$ gives the slope of the tangent drawn to the curve of intersection of the surface $z = f(x, y)$ and a plane parallel to zx-plane.

Similarly, $\frac{\partial z}{\partial y}$ gives the slope of the tangent drawn to the curve of intersection of the surface $z = f(x, y)$ and a plane parallel to yz-plane.

3.4. PARTIAL DERIVATIVES OF HIGHER ORDER

Since the first order partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are themselves functions of x and y, they can be further differentiated partially w.r.t. x as well as y. These are called second order partial derivatives of z. The usual notations for these second order partial derivatives are :

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \text{ or } f_{xx}; \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} \text{ or } f_{yy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} \text{ or } f_{xy}; \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} \text{ or } f_{yx}$$

In general, $$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} \text{ or } f_{xy} = f_{yx}.$$

Note 1. If $z = f(x)$, a function of single independent variable x, we get $\frac{dz}{dx}$

If $z = f(x, y)$, a function of two independent variables x and y, we get $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Similarly, for a function of more than two independent variables $x_1, x_2, \ldots\ldots, x_n$, we get $\frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, \ldots\ldots, \frac{\partial z}{\partial x_n}$.

Note 2. (*i*) If $z = u + v$, where $u = f(x, y)$, $v = \phi(x, y)$ then z is a function of x and y.

$$\therefore \qquad \frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}; \quad \frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

(*ii*) If $z = uv$, where $u = f(x, y)$, $v = \phi(x, y)$ then $\dfrac{\partial z}{\partial x} = \dfrac{\partial}{\partial x}(uv) = u\dfrac{\partial v}{\partial x} + v\dfrac{\partial u}{\partial x}$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(uv) = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}$$

(*iii*) If $z = \dfrac{u}{v}$, where $u = f(x, y)$, $v = \phi(x, y)$ then $\dfrac{\partial z}{\partial x} = \dfrac{\partial}{\partial x}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{\partial u}{\partial x} - u\dfrac{\partial v}{\partial x}}{v^2}$,

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{u}{v}\right) = \frac{v\dfrac{\partial u}{\partial y} - u\dfrac{\partial v}{\partial y}}{v^2}$$

(*iv*) If $z = f(u)$, where $u = \phi(x, y)$ then $\dfrac{\partial z}{\partial x} = \dfrac{dz}{du} \cdot \dfrac{\partial u}{\partial x}$; $\dfrac{\partial z}{\partial y} = \dfrac{dz}{du} \cdot \dfrac{\partial u}{\partial y}$.

Example 7. (*a*) *If* $z(x+y) = x^2 + y^2$, *show that* $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$.

(Mysore, 199[illegible])

(*b*) *If* $z = e^{ax+by} f(ax - by)$ *find* $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y}$ (P.T.U., May 200[illegible])

Sol. (*a*) $z = \frac{x^2 + y^2}{x + y}$ [z is symmetrical w.r.t. x and y]

$$\frac{\partial z}{\partial x} = \frac{(x+y)\frac{\partial}{\partial x}(x^2+y^2) - (x^2+y^2)\frac{\partial}{\partial x}(x+y)}{(x+y)^2}$$

$$= \frac{(x+y).2x - (x^2+y^2).1}{(x+y)^2} = \frac{x^2 + 2xy - y^2}{(x+y)^2}$$

Similarly, $\frac{\partial z}{\partial y} = \frac{y^2 + 2xy - x^2}{(x+y)^2}$

Now $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \left[\frac{2x^2 - 2y^2}{(x+y)^2}\right]^2 = \frac{4(x+y)^2(x-y)^2}{(x+y)^4} = \frac{4(x-y)^2}{(x+y)^2}$

$$4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = 4\left[1 - \frac{x^2 + 2xy - y^2}{(x+y)^2} - \frac{y^2 + 2xy - x^2}{(x+y)^2}\right]$$

$$= 4\left[\frac{x^2 + 2xy + y^2 - x^2 - 2xy + y^2 - y^2 - 2xy + x^2}{(x+y)^2}\right]$$

$$= \frac{4(x^2 - 2xy + y^2)}{(x+y)^2} = \frac{4(x-y)^2}{(x+y)^2}$$

$$\therefore \quad \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right).$$

(b)
$$z = e^{ax+by} f(ax - by)$$

$$\frac{\partial z}{\partial x} = e^{ax+by} f'(ax - by)\, a + ae^{ax+by} f(ax - by)$$

$$\frac{\partial z}{\partial y} = e^{ax+by} f'(ax - by)\,(-b) + be^{ax+by} f(ax - by)$$

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = abe^{ax+by} f'(ax - by) + abe^{ax+by} f(ax - by)$$
$$- ab\, e^{ax+by} f'(ax - by) + ab\, e^{ax+by} f(ax - by)$$
$$= abe^{ax+by}\, [f'(ax - by) + f(ax - by) - f'(ax - by) + f(ax - by)]$$
$$= ab\, e^{ax+by} \,.\, 2f(ax - by)$$
$$= 2ab\, e^{ax+by} f(ax - by)$$
$$= 2ab.\, z.$$

Example 8. (a) *If* $z = x\,f(x+y) + y\,g(x+y)$, *show that* $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0.$

(b) *If* $z = f(x + ay) + \phi(x - ay)$, *prove that* $\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$.

Sol. (a)
$$z = x\, f(x+y) + y\, g(x+y)$$

$$\frac{\partial z}{\partial x} = x\, f'(x+y) + f(x+y) + y\, g'(x+y)$$

$$\frac{\partial^2 z}{\partial x^2} = x\, f''(x+y) + f'(x+y) + f'(x+y) + y\, g''(x+y)$$
$$= x\, f''(x+y) + 2f'(x+y) + y\, g''(x+y)$$

$$\frac{\partial z}{\partial y} = x\, f'(x+y) + y\, g'(x+y) + g(x+y)$$

$$\frac{\partial^2 z}{\partial y^2} = x\, f''(x+y) + y\, g''(x+y) + 1.g'(x+y) + g'(x+y)$$
$$= x\, f''(x+y) + 2g'(x+y) + y\, g''(x+y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\, [x\, f'(x+y) + y\, g'(x+y) + g(x+y)]$$
$$= x\, f''(x+y) + f'(x+y) + y\, g''(x+y) + g'(x+y)$$

Now $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = x f''(x+y) + 2f'(x+y) + yg''(x+y)$

$$- 2x f''(x+y) - 2f'(x+y) - 2yg''(x+y) - 2g'(x+y)$$
$$+ x f''(x+y) + 2g'(x+y) + y g''(x+y) = 0$$

Hence $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = 0$

(b)
$$z = f(x+ay) + \phi(x-ay)$$
$$\frac{\partial z}{\partial x} = f'(x+ay) + \phi'(x-ay)$$
$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) + \phi''(x-ay)$$
$$\frac{\partial z}{\partial y} = a f'(x+ay) - a\phi'(x-ay)$$
$$\frac{\partial^2 z}{\partial y^2} = a^2 f''(x+ay) + a^2\phi''(x-ay)$$
$$= a^2 [f''(x+ay) + \phi''(x-ay)] = a^2 \frac{\partial^2 z}{\partial x^2}.$$

Example 9. (a) *Prove that if* $f(x, y) = \dfrac{1}{\sqrt{y}} . e^{-\frac{(x-a)^2}{4y}}$ *, then* $f_{xy} = f_{yx}$.

(b) *If* $v = \dfrac{1}{\sqrt{t}} e^{-\frac{x^2}{4a^2 t}}$, *prove that* $\dfrac{\partial v}{\partial t} = a^2 \dfrac{\partial^2 v}{\partial x^2}$. (Nagpur 1999)

Sol. (a) $f(x, y) = \dfrac{1}{\sqrt{y}} . e^{-\frac{(x-a)^2}{4y}} = y^{-\frac{1}{2}} e^{-\frac{(x-a)^2}{4y}}$

$$f_x = \frac{\partial f}{\partial x} = y^{-\frac{1}{2}} . e^{-\frac{(x-a)^2}{4y}} . \frac{\partial}{\partial x}\left[-\frac{(x-a)^2}{4y}\right]$$
$$= y^{-\frac{1}{2}} . e^{-\frac{(x-a)^2}{4y}} . \left[-\frac{2(x-a)}{4y}\right] = -\frac{1}{2} y^{-\frac{3}{2}} (x-a) e^{-\frac{(x-a)^2}{4y}}$$
$$f_y = \frac{\partial f}{\partial y} = -\frac{1}{2} y^{-\frac{3}{2}} . e^{-\frac{(x-a)^2}{4y}} + y^{-\frac{1}{2}} . e^{-\frac{(x-a)^2}{4y}} . \frac{\partial}{\partial y}\left[-\frac{(x-a)^2}{4y}\right]$$
$$= e^{-\frac{(x-a)^2}{4y}} \left[-\frac{1}{2} y^{-\frac{3}{2}} + y^{-\frac{1}{2}} . \frac{(x-a)^2}{4y^2}\right] = \frac{1}{4} y^{-\frac{3}{2}} e^{-\frac{(x-a)^2}{4y}} [-2 + y^{-1}(x-a)^2]$$
$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{1}{4} y^{-\frac{3}{2}} \left\{ e^{-\frac{(x-a)^2}{4y}} \cdot \frac{\partial}{\partial x}\left[-\frac{(x-a)^2}{4y}\right] \cdot [-2 + y^{-1}(x-a)^2] + e^{-\frac{(x-a)^2}{4y}} \cdot 2y^{-1}(x-a) \right\}$$

$$= \frac{1}{4} y^{-\frac{3}{2}} \cdot e^{-\frac{(x-a)^2}{4y}} \left\{ -\frac{2(x-a)}{4y} [-2 + y^{-1}(x-a)^2] + 2y^{-1}(x-a) \right\}$$

$$= \frac{1}{4} y^{-\frac{3}{2}} \cdot e^{-\frac{(x-a)^2}{4y}} \cdot \frac{x-a}{y} \left\{ -\frac{1}{2} [-2 + y^{-1}(x-a)^2] + 2 \right\}$$

$$= \frac{1}{4} y^{-\frac{5}{2}} (x-a) e^{-\frac{(x-a)^2}{4y}} \left[3 - \frac{(x-a)^2}{2y} \right]$$

$$f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\frac{1}{2}(x-a)\left[-\frac{3}{2} y^{-\frac{5}{2}} \cdot e^{-\frac{(x-a)^2}{4y}} + y^{-\frac{3}{2}} \cdot e^{-\frac{(x-a)^2}{4y}} \cdot \frac{(x-a)^2}{4y^2} \right]$$

$$= -\frac{1}{4}(x-a) y^{-\frac{5}{2}} \cdot e^{-\frac{(x-a)^2}{4y}} \left[-3 + \frac{(x-a)^2}{2y} \right]$$

$$= \frac{1}{4} y^{-\frac{5}{2}} (x-a) e^{-\frac{(x-a)^2}{4y}} \left[3 - \frac{(x-a)^2}{2y} \right]$$

$\therefore \quad f_{xy} = f_{yx}$.

(b)
$$v = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4a^2 t}}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{2t^{3/2}} e^{-\frac{x^2}{4a^2 t}} + \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4a^2 t}} \left(-\frac{x^2}{4a^2}\right)\left(-\frac{1}{t^2}\right)$$

$$= \frac{1}{2t^{3/2}} e^{-\frac{x^2}{4a^2 t}} \left[-1 + \frac{x^2}{2a^2 t} \right] \qquad \ldots(1)$$

$$\frac{\partial v}{\partial x} = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4a^2 t}} \left(\frac{-2x}{4a^2 t} \right) = -\frac{1}{2a^2 t^{3/2}} x e^{-\frac{x^2}{4a^2 t}}$$

$$\frac{\partial^2 v}{\partial x^2} = -\frac{1}{2a^2 t^{3/2}} \left\{ x e^{-\frac{x^2}{4a^2 t}} \cdot \left(\frac{-2x}{4a^2 t} \right) + e^{-\frac{x^2}{4a^2 t}} \cdot 1 \right\}$$

$$= -\frac{e^{-\frac{x^2}{4a^2 t}}}{2a^2 t^{3/2}} \left\{ -\frac{x^2}{2a^2 t} + 1 \right\}$$

$$= \frac{1}{a^2} \left\{ \frac{e^{-\frac{x^2}{4a^2 t}}}{2 t^{3/2}} \left(\frac{x^2}{2a^2 t} - 1 \right) \right\}$$

$$= \frac{1}{a^2}\frac{\partial v}{\partial t}$$ [From (1)]

Hence $$\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}.$$

Example 10. *If $u = x^y$, show that* $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$ (P.T.U. May 1999)

Sol. $$u = x^y$$

Take logs of both sides

$$\log u = y \log x \quad \text{...(1)}$$

Differentiate (1) partially w.r.t. y,

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial y} = \log x \qquad \therefore \quad \frac{\partial u}{\partial y} = x^y \log x$$

Differentiate (1) partially w.r.t. x,

$$\frac{1}{u}\frac{\partial u}{\partial x} = \frac{y}{x} \qquad \therefore \quad \frac{\partial u}{\partial x} = x^y\,\frac{y}{x} = yx^{y-1}$$

Now $$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}[u \log x] = u\,.\,\frac{1}{x} + \log x\,.\,\frac{\partial u}{\partial x} = \frac{x^y}{x} + yx^{y-1}\log x$$

$$= yx^{y-1}\log x + x^y\,.\,\frac{1}{x} = x^{y-1}(y \log x + 1)$$

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x \partial y}\right) = \frac{\partial}{\partial x}\,[x^{y-1}(y \log x + 1)] \quad \text{...(2)}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{uy}{x}\right) = \frac{1}{x}\left\{u\,.\,1 + y\frac{\partial u}{\partial y}\right\}$$

$$= \frac{1}{x}\,[x^y + yx^y \log x] = x^{y-1} + yx^{y-1}\log x = x^{y-1}(y \log x + 1)$$

$$\frac{\partial^3 u}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial x}\right) = \frac{\partial}{\partial x}\,[x^{y-1}(y \log x + 1)] \quad \text{...(3)}$$

From (2) and (3), $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$.

Example 11. *Find p, q if $x = \sqrt{a}$ (sin u + cos v); $y = \sqrt{a}$ (cos u − sin v), z = 1 + sin (u − v) where p, q means $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ respectively.*

Sol. Given

$$x = \sqrt{a}\,(\sin u + \cos v)$$
$$y = \sqrt{a}\,(\cos u - \sin v)$$
$$z = 1 + \sin(u - v)$$
$$x^2 + y^2 = a[\sin^2 u + \cos^2 v + 2\sin u \cos v + \cos^2 u + \sin^2 v - 2\cos u \sin v]$$
$$= a\,[2 + 2\sin(u - v)]$$
$$= 2a\,.\,z$$

$$\therefore \qquad z = \frac{1}{2a}(x^2 + y^2)$$

$$p = \frac{\partial z}{\partial x} = \frac{2x}{2a} = \frac{x}{a}$$

$$q = \frac{\partial z}{\partial y} = \frac{2y}{2a} = \frac{y}{a}.$$

Example 12. *If* $\theta = t^n e^{-\frac{r^2}{4t}}$, *find the value of n which will make* $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$.

(P.T.U., May 2002 Calicut, 1994 ; A.M.I.E. 1990)

Sol. $$\theta = t^n e^{-\frac{r^2}{4t}}$$

$$\frac{\partial \theta}{\partial r} = t^n . e^{-\frac{r^2}{4t}} . \left(-\frac{2r}{4t}\right) = -\frac{1}{2} r t^{n-1} e^{-\frac{r^2}{4t}}$$

$$\therefore \qquad r^2 \frac{\partial \theta}{\partial r} = -\frac{1}{2} r^3 . t^{n-1} e^{-\frac{r^2}{4t}}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = -\frac{1}{2} t^{n-1}\left[3r^2 e^{-\frac{r^2}{4t}} + r^3 e^{-\frac{r^2}{4t}}\left(-\frac{2r}{4t}\right)\right] = -\frac{1}{2} t^{n-1} r^2 e^{-\frac{r^2}{4t}}\left[3 - \frac{r^2}{2t}\right]$$

$$\therefore \qquad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{1}{2} t^{n-1} e^{-\frac{r^2}{4t}}\left(\frac{r^2}{2t} - 3\right)$$

Also $$\frac{\partial \theta}{\partial t} = n t^{n-1} e^{-\frac{r^2}{4t}} + t^n e^{-\frac{r^2}{4t}} . \left(\frac{r^2}{4t^2}\right) = t^{n-1} e^{-\frac{r^2}{4t}}\left(n + \frac{r^2}{4t}\right)$$

Since $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t} \qquad \text{[given]}$$

$$\therefore \qquad \frac{1}{2} t^{n-1} e^{-\frac{r^2}{4t}}\left(\frac{r^2}{2t} - 3\right) = t^{n-1} e^{-\frac{r^2}{4t}}\left(n + \frac{r^2}{4t}\right)$$

$$\Rightarrow \qquad \frac{r^2}{4t} - \frac{3}{2} = n + \frac{r^2}{4t} \quad \therefore \quad n = -\frac{3}{2}.$$

Example 13. *If* $u = (1 - 2xy + y^2)^{-1/2}$, *prove that* most imp #

(a) $$\frac{\partial}{\partial x}\left\{(1 - x^2)\frac{\partial u}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{y^2 \frac{\partial u}{\partial y}\right\} = 0.$$ (Hamirpur, 1995)

(b) $$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = y^2 u^3.$$

Sol. (a) $$u = (1 - 2xy + y^2)^{-1/2} = V^{-1/2}, \quad \text{where } V = 1 - 2xy + y^2$$

$$\frac{\partial u}{\partial x} = -\frac{1}{2}V^{-3/2} . \frac{\partial V}{\partial x} = -\frac{1}{2}V^{-3/2}(-2y) = yV^{-3/2}$$

$$\frac{\partial^2 u}{\partial x^2} = y . \frac{\partial}{\partial x}(V^{-3/2}) = y . \left(-\frac{3}{2}\right)V^{-5/2} . \frac{\partial V}{\partial x} = -\frac{3}{2} yV^{-5/2}(-2y) = 3y^2 V^{-5/2}$$

$$\therefore \quad \frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\} = (1-x^2)\cdot\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\cdot\frac{\partial}{\partial x}(1-x^2)$$

$$= (1-x^2)\,.\,3y^2V^{-5/2} + yV^{-3/2}(-2x) = yV^{-3/2}\,[3yV^{-1}(1-x^2) - 2x] \qquad \ldots(1)$$

Also $$\frac{\partial u}{\partial y} = -\frac{1}{2}V^{-3/2}\frac{\partial V}{\partial y} = -\frac{1}{2}V^{-3/2}\,.\,(-2x+2y) = V^{-3/2}\,.\,(x-y)$$

$$\frac{\partial^2 u}{\partial y^2} = V^{-3/2}\,.\,\frac{\partial}{\partial y}(x-y) + (x-y)\,.\,\frac{y}{\partial y}(V^{-3/2})$$

$$= V^{-3/2}\,.\,(-1) + (x-y)\,.\left(-\frac{3}{2}V^{-5/2}\right).\frac{\partial V}{\partial y}$$

$$= -V^{-3/2} - \frac{3}{2}(x-y)\,V^{-5/2}\,.\,(-2x+2y) = -V^{-3/2} + 3(x-y)^2\,V^{-5/2}$$

$$\therefore \quad \frac{\partial}{\partial y}\left\{y^2\frac{\partial u}{\partial y}\right\} = y^2\,.\,\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\,.\,\frac{\partial}{\partial y}(y^2)$$

$$= y^2[-V^{-3/2} + 3(x-y)^2V^{-5/2}] + V^{-3/2}(x-y)\,.\,2y$$

$$= y\,V^{-3/2}\,[-y + 3y(x-y)^2V^{-1} + 2(x-y)]$$

$$= y\,V^{-3/2}\,[3y\,(x-y)^2V^{-1} + (2x-3y)] \qquad \ldots(2)$$

Adding (1) and (2), we have

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{y^2\frac{\partial u}{\partial y}\right\} = yV^{-3/2}\,[3yV^{-1}(1-x^2) - 2x + 3y\,(x-y)^2\,V^{-1} + 2x - 3y]$$

$$= y\,V^{-3/2}[3yV^{-1}\,(1-x^2+x^2-2xy+y^2) - 3y]$$

$$= yV^{-3/2}[3yV^{-1}(1-2xy+y^2) - 3y]$$

$$= yV^{-3/2}\,[3y-3y] \qquad \mid \because \quad V = 1-2xy+y^2$$

$$= 0.$$

(*b*) From (*a*) part $$\frac{\partial u}{\partial x} = y\,V^{-3/2}$$

and $$\frac{\partial u}{\partial y} = (x-y)\,V^{-3/2}$$

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = (xy - xy + y^2)\,V^{-3/2} = y^2\,(1-2xy+y^2)^{-3/2} = y^2u^3.$$

Hence $$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = y^2u^3.$$

Example 14. *If* $u = \log\,(x^3 + y^3 + z^3 - 3xyz)$, *show that*

(*i*) $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$$

(Rewa, 1990 ; Delhi, 1997 ; Hamirpur, 1994 S ; P.T.U., Dec. 1998, 2000 ; May 2003, Dec. 2004, Dec. 2005, May 2007)

(*ii*) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + 2\frac{\partial^2 u}{\partial y\partial z} + 2\frac{\partial^2 u}{\partial z\partial x} + 2\frac{\partial^2 u}{\partial x\partial y} = \frac{-9}{(x+y+z)^2}.$$

Sol. (*i*) $$u = \log(x^3 + y^3 + z^3 - 3xyz)$$

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}\,;\quad \frac{\partial u}{\partial y} = \frac{3y^2 - 3zx}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

Adding, $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{x^3 + y^3 + z^3 - 3xyz} = \frac{3}{x + y + z}$$

$$[\because\ x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)]$$

Now $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x + y + z}\right)$$

$$= -\frac{3}{(x + y + z)^2} - \frac{3}{(x + y + z)^2} - \frac{3}{(x + y + z)^2} = -\frac{9}{(x + y + z)^2} \qquad \ldots(1)$$

(*ii*) $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 u}{\partial z \partial x} + \frac{\partial^2 u}{\partial z \partial y} + \frac{\partial^2 u}{\partial z^2}$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + 2\frac{\partial^2 u}{\partial y \partial z} + 2\frac{\partial^2 u}{\partial z \partial x} + 2\frac{\partial^2 u}{\partial x \partial y}$$

$$\left[\because\ \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y},\ \frac{\partial^2 u}{\partial z \partial y} = \frac{\partial^2 u}{\partial y \partial z},\ \frac{\partial^2 u}{\partial x \partial z} = \frac{\partial^2 u}{\partial z \partial x}\right]$$

$$\therefore\quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + 2\frac{\partial^2 u}{\partial y \partial z} + 2\frac{\partial^2 u}{\partial z \partial x} + 2\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x + y + z)^2}$$

[from (1)]

Example 15. *If $x^x y^y z^z = c$, show that at $x = y = z$, $\dfrac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$.*

(P.T.U. 1999 ; A.M.I.E., 1996 W)

Sol. $x^x y^y z^z = c$ defines z as a function of x and y.

Taking logs of both sides $x \log x + y \log y + z \log z = \log c$

Differentiating partially w.r.t. y, we have

$$y \cdot \frac{1}{y} + 1 \cdot \log y + z \cdot \frac{1}{z} \cdot \frac{\partial z}{\partial y} + 1 \cdot \log z \cdot \frac{\partial z}{\partial y} = 0$$

or $$1 + \log y + (1 + \log z)\frac{\partial z}{\partial y} = 0 \qquad \ldots(1)$$

or
$$\left.\begin{aligned}\frac{\partial z}{\partial y} &= -\frac{1+\log y}{1+\log z}\\ \frac{\partial z}{\partial x} &= -\frac{1+\log x}{1+\log z}\end{aligned}\right\} \quad \ldots(2)$$
Similarly,

Differentiating (1) partially w.r.t. x, we have

$$\left(\frac{1}{z}\frac{\partial z}{\partial x}\right)\frac{\partial z}{\partial y} + (1+\log z)\frac{\partial^2 z}{\partial x \partial y} = 0 \quad \text{or} \quad \frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{z(1+\log z)}\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} \quad \ldots(3)$$

When $x = y = z$

From (2), $\quad \dfrac{\partial z}{\partial y} = -1, \dfrac{\partial z}{\partial x} = -1$

From (3), $\dfrac{\partial^2 z}{\partial x \partial y} = -\dfrac{1}{x(1+\log x)}(-1)(-1) = -\dfrac{1}{x(\log e + \log x)} = -\dfrac{1}{x(\log ex)} = -(x \log ex)^{-1}$.

Example 16. *If $u = f(r)$, where $r^2 = x^2 + y^2$, prove that* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f''(r) + \dfrac{1}{r} f'(r)$.

(Andhra, 1990 ; Rewa, 1990 ; Mysore, 1994 ; Mangalore, 1997)

Sol. $\quad r^2 = x^2 + y^2 \quad \ldots(1)$

Differentiating partially w.r.t. x, we get $2r\dfrac{\partial r}{\partial x} = 2x \quad$ or $\quad \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

Similarly, $\quad \dfrac{\partial r}{\partial y} = \dfrac{y}{r}$

Now $\quad u = f(r)$

$\therefore \quad \dfrac{\partial u}{\partial x} = f'(r)\cdot\dfrac{\partial r}{\partial x} = \dfrac{x}{r} f'(r)$

Differentiating again w.r.t. x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{r}f'(r) + x\cdot\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right)f'(r) + \frac{x}{r}f''(r)\cdot\frac{\partial r}{\partial x}$$

$$\left[\because \frac{\partial}{\partial x}(uvw) = vw\frac{\partial}{\partial x}(u) + uw\frac{\partial}{\partial x}(v) + uv\frac{\partial}{\partial x}(w)\right]$$

$$= \frac{1}{r}f'(r) - \frac{x}{r^2}\cdot\frac{x}{r}f'(r) + \frac{x}{r}\cdot f''(r)\cdot\frac{x}{r} = \frac{1}{r}f'(r) - \frac{x^2}{r^3}f'(r) + \frac{x^2}{r^2}f''(r)$$

$$= \frac{r^2 - x^2}{r^3}f'(r) + \frac{x^2}{r^2}f''(r) = \frac{y^2}{r^3}f'(r) + \frac{x^2}{r^2}f''(r) \quad | \text{ using } (1)$$

Similarly, $\quad \dfrac{\partial^2 u}{\partial y^2} = \dfrac{x^2}{r^3}f'(r) + \dfrac{y^2}{r^2}f''(r)$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{x^2 + y^2}{r^3}f'(r) + \frac{x^2 + y^2}{r^2}f''(r) = \frac{r^2}{r^3}f'(r) + \frac{r^2}{r^2}f''(r) = f''(r) + \frac{1}{r}f'(r).$$

Example 17. (*a*) *If $V = f(r)$ and $r^2 = x^2 + y^2 + z^2$ prove that* $V_{xx} + V_{yy} + V_{zz} = f''(r) + \dfrac{2}{r}f'(r)$.

(*b*) *If $V = (x^2 + y^2 + z^2)^{-1/2}$, prove that* $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = 0$.

(c) If $V = r^m$ where $r^2 = x^2 + y^2 + z^2$, show that $V_{xx} + V_{yy} + V_{zz} = m(m+1)\, r^{m-2}$.

(P.T.U. Dec. 2006

Sol. (a) $V = f(r),\ r^2 = x^2 + y^2 + z^2$

$$\therefore \quad 2r\frac{\partial r}{\partial x} = 2x \quad \therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r}. \text{ Similarly, } \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$V_x = \frac{\partial V}{\partial x} = f'(r)\,.\,\frac{\partial r}{\partial x} = \frac{x}{r}\,f'(r)$$

$$V_{xx} = x\,.\,\frac{1}{r}\,.\,f''(r)\,\frac{\partial r}{\partial x} + x f'(r)\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial x} + \frac{f'(r)}{r}$$

$$= \frac{x}{r}\,f''(r)\,.\,\frac{x}{r} - f'(r)\,\frac{x}{r^2}\,.\,\frac{x}{r} + \frac{f'(r)}{r}$$

$$= \frac{x^2}{r^2}\,f''(r) + f'(r)\left[\frac{1}{r} - \frac{x^2}{r^3}\right]$$

$$= \frac{x^2}{r^2}\,f''(r) + \frac{1}{r^3}\,(r^2 - x^2)\,f'(r)$$

Similarly, $$V_{yy} = \frac{y^2}{r^2}\,f''(r) + \frac{r^2 - y^2}{r^3}\,f'(r)$$

$$V_{zz} = \frac{z^2}{r^2}\,f''(r) + \frac{r^2 - z^2}{r^3}\,f'(r)$$

Adding, $$V_{xx} + V_{yy} + V_{zz} = \frac{x^2 + y^2 + z^2}{r^2}\,f''(r) + \frac{3r^2 - x^2 - y^2 - z^2}{r^3}\,f'(r)$$

$$= \frac{r^2}{r^2}\,f''(r) + \frac{3r^2 - r^2}{r^3}\,f'(r)$$

$$= f''(r) + \frac{2}{r}\,f'(r)$$

(b) $$V = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$$

$$\frac{\partial V}{\partial x} = -\frac{1}{2}\,(x^2 + y^2 + z^2)^{-3/2}\,(2x) = -x\,(x^2 + y^2 + z^2)^{-3/2}$$

Similarly, $$\frac{\partial V}{\partial y} = -y(x^2 + y^2 + z^2)^{-3/2}, \quad \frac{\partial V}{\partial z} = -z\,(x^2 + y^2 + z^2)^{-3/2}$$

$$\frac{\partial^2 V}{\partial x^2} = -x\left(\frac{-3}{2}\right)(x^2 + y^2 + z^2)^{-5/2}\,(2x) - (x^2 + y^2 + z^2)^{-3/2}$$

$$= 3x^2\,(x^2 + y^2 + z^2)^{-5/2} - (x^2 + y^2 + z^2)^{-3/2}$$

and, $$\frac{\partial^2 V}{\partial y^2} = 3y^2\,(x^2 + y^2 + z^2)^{-5/2} - (x^2 + y^2 + z^2)^{-3/2}$$

$$\frac{\partial^2 V}{\partial z^2} = 3z^2\,(x^2 + y^2 + z^2)^{-5/2} - (x^2 + y^2 + z^2)^{-3/2}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 3(x^2+y^2+z^2)(x^2+y^2+z^2)^{-5/2} - 3(x^2+y^2+z^2)^{-3/2}$$
$$= 3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)^{-3/2} = 0$$

Hence $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$

(c) $V = r^m$, $\quad r^2 = x^2 + y^2 + z^2 \quad \therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial y} = \frac{z}{r}$

$$V_x = m\, r^{m-1} \frac{\partial r}{\partial x} = m\, r^{m-1} \cdot \frac{x}{r} = m\, r^{m-2} x$$
$$V_{xx} = m \left\{ r^{m-2} \cdot 1 + x \cdot (m-2) r^{m-3} \cdot \frac{\partial r}{\partial x} \right\}$$
$$= m\, r^{m-2} + m(m-2)\, r^{m-3} \cdot \frac{x^2}{r}$$
$$V_{xx} = m\, r^{m-2} + m(m-2)\, r^{m-4} \cdot x^2$$

Similarly,
$$V_{yy} = m\, r^{m-2} + m(m-2)\, r^{m-4} y^2$$
$$V_{zz} = m\, r^{m-2} + m(m-2)\, r^{m-4} z^2$$
$$V_{xx} + V_{yy} + V_{zz} = 3m\, r^{m-2} + m(m-2)\, r^{m-4}(x^2+y^2+z^2)$$
$$= 3m\, r^{m-2} + m(m-2)\, r^{m-4} r^2$$
$$= 3m\, r^{m-2} + m(m-2)\, r^{m-2}$$
$$= m(m+1)\, r^{m-2}$$

Hence $V_{xx} + V_{yy} + V_{zz} = m(m+1)\, r^{m-2}$.

Note. (*b*) part can be deduced from (*c*) part by putting $m = -1$ and (*c*) part can be deduced from (*a*) part by putting $f(r) = r^m$.

Example 18. *If* $x = r\cos\theta$, $y = r\sin\theta$, *prove that*

(*i*) $\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$ (*ii*) $\frac{1}{r} \cdot \frac{\partial x}{\partial \theta} = r \frac{\partial \theta}{\partial x}$ (*iii*) $\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0.$

(*iv*) $\frac{\partial^2 r}{\partial x^2} \cdot \frac{\partial^2 r}{\partial y^2} = \left(\frac{\partial^2 r}{\partial x \partial y}\right)^2$ (*v*) $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$. (Mysore 1994)

Sol. (*i*) $\frac{\partial r}{\partial x}$ means $\left(\frac{\partial r}{\partial x}\right)_y$ = the partial derivative of r w.r.t. x, treating y as constant.

$\therefore$ We express r in terms of x and y.

Squaring and adding the given relations, $r^2 = x^2 + y^2$

Differentiating partially w.r.t. x, we get $2r \frac{\partial r}{\partial x} = 2x$ or $\frac{\partial r}{\partial x} = \frac{x}{r}$

$\frac{\partial x}{\partial r}$ means $\left(\frac{\partial x}{\partial r}\right)_\theta$ = the partial derivative of x w.r.t. r treating θ as constant.

$\therefore$ we express x in terms of r and θ.

Thus, $\quad x = r\cos\theta \quad$ (given)

$$\frac{\partial x}{\partial r} = \cos\theta = \frac{x}{r} \qquad \left(\because \cos\theta = \frac{x}{r}\right)$$

$$\therefore \qquad \frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$$

(ii) Expressing x in terms of r and θ, we have $x = r\cos\theta$

$$\Rightarrow \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta = -y \quad \Rightarrow \quad \frac{1}{r}\frac{\partial x}{\partial \theta} = -\frac{y}{r}$$

Expressing θ in terms of x and y, we have $\tan\theta = \frac{y}{x}$ or $\theta = \tan^{-1}\frac{y}{x}$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = \frac{-y}{r^2(\cos^2\theta+\sin^2\theta)} = -\frac{y}{r^2}$$

$$\Rightarrow \qquad r\frac{\partial \theta}{\partial x} = -\frac{y}{r} \qquad \therefore \quad \frac{1}{r}\cdot\frac{\partial x}{\partial \theta} = r\frac{\partial \theta}{\partial x}.$$

(iii) Expressing θ in terms of x and y, we have $\tan\theta = \frac{y}{x}$ or $\theta = \tan^{-1}\frac{y}{x}$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -y(x^2+y^2)^{-1}$$

$$\frac{\partial^2 \theta}{\partial x^2} = y(x^2+y^2)^{-2}\,.\,2x = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x} = \frac{x}{x^2+y^2} = x(x^2+y^2)^{-1}$$

$$\frac{\partial^2 \theta}{\partial y^2} = -x(x^2+y^2)^{-2}\,.\,2y = -\frac{2xy}{(x^2+y^2)^2}$$

$$\therefore \qquad \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0.$$

(iv) from (i) part $r^2 = x^2 + y^2 \quad \therefore \quad r = (x^2+y^2)^{1/2}$

$$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2+y^2)^{-1/2}\,.\,2x = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial r}{\partial y} = \frac{1}{2}(x^2+y^2)^{-1/2}\,.\,2y = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{\sqrt{x^2+y^2}\,.1 - x\,.\frac{1}{2\sqrt{x^2+y^2}}\,.\,2x}{x^2+y^2} = \frac{x^2+y^2-x^2}{(x^2+y^2)^{3/2}} = \frac{y^2}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial^2 r}{\partial y^2} = \frac{\sqrt{x^2+y^2}\,.1 - y\,.\frac{1}{2\sqrt{x^2+y^2}}\,.\,2y}{x^2+y^2} = \frac{x^2+y^2-y^2}{(x^2+y^2)^{3/2}} = \frac{x^2}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial^2 r}{\partial x \partial y} = y\,.\,\frac{-1}{2}(x^2+y^2)^{-3/2}\,.\,2x = \frac{-xy}{(x^2+y^2)^{3/2}}$$

Now $$\frac{\partial^2 r}{\partial x^2}\cdot\frac{\partial^2 r}{\partial y^2}=\frac{y^2}{(x^2+y^2)^{3/2}}\cdot\frac{x^2}{(x^2+y^2)^{3/2}}=\frac{x^2y^2}{(x^2+y^2)^{3/2}}=\left(\frac{\partial^2 r}{\partial x\partial y}\right)^2$$

(v) And $$\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}=\frac{y^2}{(x^2+y^2)^{3/2}}+\frac{x^2}{(x^2+y^2)^{3/2}}=\frac{x^2+y^2}{(x^2+y^2)^{3/2}}=\frac{1}{(x^2+y^2)^{1/2}}=\frac{1}{r}$$

$$\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2=\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}=\frac{x^2+y^2}{x^2+y^2}=1$$

$\therefore$ $$\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}=\frac{1}{r}\cdot 1=\frac{1}{r}\left\{\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right\}.$$

Example 19. *If $x=e^{r\cos\theta}\cos(r\sin\theta)$ and $y=e^{r\cos\theta}\sin(r\sin\theta)$, prove that*

$$\frac{\partial x}{\partial r}=\frac{1}{r}\cdot\frac{\partial y}{\partial\theta},\ \frac{\partial y}{\partial r}=-\frac{1}{r}\cdot\frac{\partial x}{\partial\theta}.$$

Hence deduce that $$\frac{\partial^2 x}{\partial r^2}+\frac{1}{r}\frac{\partial x}{\partial r}+\frac{1}{r^2}\frac{\partial^2 x}{\partial\theta^2}=0.$$

Sol. $$x=e^{r\cos\theta}\cos(r\sin\theta)$$

$\therefore$ $$\frac{\partial x}{\partial r}=e^{r\cos\theta}\cdot\cos\theta\cdot\cos(r\sin\theta)-e^{r\cos\theta}\sin(r\sin\theta)\cdot\sin\theta$$
$$=e^{r\cos\theta}[\cos\theta\cos(r\sin\theta)-\sin\theta\sin(r\sin\theta)]$$
$$=e^{r\cos\theta}\cos(\theta+r\sin\theta)$$
$$\frac{\partial x}{\partial\theta}=e^{r\cos\theta}\cdot(-r\sin\theta)\cdot\cos(r\sin\theta)-e^{r\cos\theta}\sin(r\sin\theta)\cdot r\cos\theta$$
$$=-re^{r\cos\theta}[\sin\theta\cos(r\sin\theta)+\cos\theta\sin(r\sin\theta)]$$
$$=-re^{r\cos\theta}\sin(\theta+r\sin\theta)$$

Also $$y=e^{r\cos\theta}\sin(r\sin\theta)$$

$\therefore$ $$\frac{\partial y}{\partial r}=e^{r\cos\theta}\cdot\cos\theta\cdot\sin(r\sin\theta)+e^{r\cos\theta}\cdot\cos(r\sin\theta)\sin\theta$$
$$=e^{r\cos\theta}[\sin\theta\cos(r\sin\theta)+\cos\theta\sin(r\sin\theta)]$$
$$=e^{r\cos\theta}\sin(\theta+r\sin\theta)$$
$$\frac{\partial y}{\partial\theta}=e^{r\cos\theta}(-r\sin\theta)\sin(r\sin\theta)+e^{r\cos\theta}\cos(r\sin\theta)\times r\cos\theta$$
$$=re^{r\cos\theta}[\cos\theta\cos(r\sin\theta)-\sin\theta\sin(r\sin\theta)]$$
$$=re^{r\cos\theta}\cos(\theta+r\sin\theta)$$

From (1) and (4), $$\frac{\partial x}{\partial r}=\frac{1}{r}\cdot\frac{\partial y}{\partial\theta}$$

From (2) and (3), $$\frac{\partial y}{\partial r}=-\frac{1}{r}\cdot\frac{\partial x}{\partial\theta}$$

From (5), $$\frac{\partial^2 x}{\partial r^2}=-\frac{1}{r^2}\cdot\frac{\partial y}{\partial\theta}+\frac{1}{r}\cdot\frac{\partial^2 y}{\partial r\partial\theta}$$

From (6),
$$\frac{\partial x}{\partial \theta} = -r\frac{\partial y}{\partial r}$$

$$\therefore \quad \frac{\partial^2 x}{\partial \theta^2} = -r\frac{\partial^2 y}{\partial \theta \partial r} = -r\frac{\partial^2 y}{\partial r \partial \theta} \qquad \ldots(8)$$

$$\therefore \quad \frac{\partial^2 x}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial x}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 x}{\partial \theta^2} = -\frac{1}{r^2}\cdot\frac{\partial y}{\partial \theta} + \frac{1}{r}\cdot\frac{\partial^2 y}{\partial r\partial\theta} + \frac{1}{r^2}\cdot\frac{\partial y}{\partial \theta} - \frac{1}{r}\cdot\frac{\partial^2 y}{\partial r\partial\theta} = 0. \text{ (Using 5, 7, 8)}$$

Example 20. *If* $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, *prove that*

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right).$$

Imp ques!

(P.T.U. Dec., 2003 ; A.M.I.E. 1997)

Sol. Given
$$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1 \qquad \ldots(1)$$

or
$$x^2(a^2+u)^{-1} + y^2(b^2+u)^{-1} + z^2(c^2+u)^{-1} = 1$$

Differentiating partially w.r.t. x, we have

$$2x(a^2+u)^{-1} - x^2(a^2+u)^{-2}\cdot\frac{\partial u}{\partial x} - y^2(b^2+u)^{-2}\cdot\frac{\partial u}{\partial x} - z^2(c^2+u)^{-2}\cdot\frac{\partial u}{\partial x} = 0$$

or
$$\frac{2x}{a^2+u} = \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]\frac{\partial u}{\partial x}$$

or
$$\frac{2x}{a^2+u} = V\frac{\partial u}{\partial x} \text{ where } V = \frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}$$

or
$$\frac{\partial u}{\partial x} = \frac{2x}{V(a^2+u)}$$

$$\frac{2x}{a^2+u} = \left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]$$

Similarly,
$$\frac{\partial u}{\partial y} = \frac{2y}{V(b^2+u)} \quad \text{and} \quad \frac{\partial u}{\partial z} = \frac{2z}{V(c^2+u)}$$

$$\therefore \quad \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = \frac{4}{V^2}\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]$$

$$\frac{4x^2}{V^2(a^2+u)^2} + \frac{4y^2}{V^2(b^2+u)^2} + \frac{z^2}{V^2(c^2+u)^2}$$

$$= \frac{4}{V^2}(V) = \frac{4}{V} \qquad \ldots(2)$$

$$= \frac{4}{V^2}[V]$$

Now,
$$2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right) = 2\left[\frac{2x^2}{V(a^2+u)} + \frac{2y^2}{V(b^2+u)} + \frac{2z^2}{V(c^2+u)}\right]$$

$$= \frac{4}{V}\left[\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u}\right]$$

$$= \frac{4}{V}(1) \qquad \text{[Using (1)]}$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \qquad \text{[Using (2)]}$$

Example 21. *If $x^2 = au + bv$, $y^2 = au - bv$, prove that* $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u$

Sol. As $$x^2 = au + bv \quad ...(1)$$
$$y^2 = au - bv \quad ...(2)$$

Add and subtract, we get $u = \frac{x^2 + y^2}{2a}$, $v = \frac{x^2 - y^2}{2b}$

$$\left(\frac{\partial u}{\partial x}\right)_y = \frac{x}{a}, \left(\frac{\partial v}{\partial y}\right)_x = -\frac{y}{b}$$

from (1), $$2x\left(\frac{\partial x}{\partial u}\right)_v = a \quad \therefore \quad \left(\frac{\partial x}{\partial u}\right)_v = \frac{a}{2x}$$

from (2), $$2y\left(\frac{\partial y}{\partial v}\right)_u = -b \quad \therefore \quad \left(\frac{\partial y}{\partial v}\right)_u = -\frac{b}{2y}$$

$$\text{L.H.S.} = \left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v \frac{x}{a} \cdot \frac{a}{2x} = \frac{1}{2}$$

$$\text{R.H.S.} = \left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u = \left(-\frac{y}{b}\right)\left(-\frac{b}{2y}\right) = \frac{1}{2}$$

Hence $$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u$$

Example 22. *If $u = lx + my$, $v = mx - ly$, show that*

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{l^2}{l^2 + m^2}, \left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u = \frac{l^2 + m^2}{l^2}$$

(Marathwada, 1990)

Sol. Given $$u = lx + my \quad ...(1)$$
$$v = mx - ly \quad ...(2)$$

(*i*) $\left(\frac{\partial u}{\partial x}\right)_y$ = The partial derivative of u w.r.t. x keeping y constant.

$\therefore$ We need a relation expressing u as a function of x and y.

From (1), $$\left(\frac{\partial u}{\partial x}\right)_y = l$$

$\left(\frac{\partial x}{\partial u}\right)_v$ = The partial derivative of x w.r.t. u keeping v constant.

$\therefore$ We need a relation expressing x as a function of u and v.

Eliminating y between (1) and (2) by multiplying (1) by l, (2) by m and adding the products, we have

$$lu + mv = (l^2 + m^2)x \quad \text{or} \quad x = \frac{lu + mv}{l^2 + m^2}$$

$$\therefore \quad \left(\frac{\partial x}{\partial u}\right)_v = \frac{l}{l^2 + m^2}$$

Hence $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{l^2}{l^2+m^2}$

(ii) $\left(\frac{\partial y}{\partial v}\right)_x$ = the partial derivative of y w.r.t. v keeping x constant.

$\therefore$ We need a relation expressing y as a function of v and x.

From (2), $y = \frac{mx - v}{l}$ $\therefore$ $\left(\frac{\partial y}{\partial v}\right)_x = -\frac{1}{l}$

Also $\left(\frac{\partial v}{\partial y}\right)_u$ = partial derivative of v w.r.t. y keeping u constant.

$\therefore$ We need a relation expressing v as a function of y and u.

Eliminating x between (1) and (2), we have $v = \frac{mu - (l^2+m^2)y}{l}$

$$\therefore \quad \left(\frac{\partial v}{\partial y}\right)_u = -\frac{l^2+m^2}{l}$$

Hence $\left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u = \left(-\frac{1}{l}\right)\left(-\frac{l^2+m^2}{l}\right) = \frac{l^2+m^2}{l^2}$.

Example 23. *If* $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$, *find the value of* $\frac{\partial^2 u}{\partial x\, \partial y}$.

(A.M.I.E. 1990, Mysore 1997, Marathwada 1994)

Sol.
$$u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$$

$$\frac{\partial u}{\partial y} = x^2 \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) - y^2 \frac{1}{1+\frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right) - \tan^{-1}\frac{x}{y} \cdot 2y$$

$$= \frac{x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y}$$

$$= \frac{x(x^2+y^2)}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y}$$

$$= x - 2y\tan^{-1}\frac{x}{y}$$

$$\frac{\partial^2 u}{\partial x\, \partial y} = 1 - 2y\frac{1}{1+\frac{x^2}{y^2}}\left(\frac{1}{y}\right)$$

$$= 1 - \frac{2y^2}{y^2+x^2} = \frac{y^2+x^2-2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$$

Hence $\frac{\partial^2 u}{\partial x\, \partial y} = \frac{x^2-y^2}{x^2+y^2}$.

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Example 24. *If* $u = f(ax^2 + 2hxy + by^2)$, $v = \phi\,(ax^2 + 2hxy + by^2)$, *prove that*

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$$\frac{\partial}{\partial y}\left(u\,\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(u\,\frac{\partial v}{\partial y}\right).$$

Sol. Let $z = ax^2 + 2hxy + by^2$ ...(1)

$\therefore$ $u = f(z),\ v = \phi(z)$ ...(2)

We have to prove $\dfrac{\partial}{\partial y}\left(u\,\dfrac{\partial v}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(u\,\dfrac{\partial v}{\partial y}\right)$

i.e., to prove $u\,\dfrac{\partial^2 v}{\partial y\,\partial x} + \dfrac{\partial u}{\partial y}\cdot\dfrac{\partial v}{\partial x} = u\cdot\dfrac{\partial^2 v}{\partial x\,\partial y} + \dfrac{\partial u}{\partial x}\cdot\dfrac{\partial v}{\partial y}$

As $\dfrac{\partial^2 v}{\partial y\,\partial x} = \dfrac{\partial^2 v}{\partial x\,\partial y}$

$\therefore$ We have to prove only $\dfrac{\partial u}{\partial y}\cdot\dfrac{\partial v}{\partial x} = \dfrac{\partial u}{\partial x}\cdot\dfrac{\partial v}{\partial y}$

from (2), $\dfrac{\partial u}{\partial x} = f'(z)\,\dfrac{\partial z}{\partial x};\ \dfrac{\partial u}{\partial y} = f'(z)\,\dfrac{\partial z}{\partial y}$

and $\dfrac{\partial v}{\partial x} = \phi'(z)\,\dfrac{\partial z}{\partial x};\ \dfrac{\partial v}{\partial y} = \phi'(z)\,\dfrac{\partial z}{\partial y}$

from (1), $= \dfrac{\partial z}{\partial x} = 2ax + 2hy$

$$\frac{\partial z}{\partial y} = 2hx + 2by$$

$\therefore$
$$\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x} = f'(z)\,(2hx + 2by)\,\phi'(z)\,(2ax + 2hy)$$
$$= 4\,f'(z)\,\phi'(z)\,(hx + by)\,(ax + hy)$$

And
$$\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} = f'(z)\,(2ax + 2hy).\,\phi'(z)\,(2hx + 2by)$$
$$= 4\,f'(z)\,\phi'(z)\,(hx + by)\,(ax + hy)$$

Hence
$$\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}$$

$\therefore$
$$\frac{\partial}{\partial y}\left(u\,\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(u\,\frac{\partial v}{\partial y}\right).$$

TEST YOUR KNOWLEDGE

1. Find the first order partial derivatives of the following functions :
   (*i*) $u = y^x$ (*ii*) $u = \log(x^2 + y^2)$
   (*iii*) $u = x^2 \sin\dfrac{y}{x}$ (*iv*) $u = \dfrac{x}{y}\tan^{-1}\left(\dfrac{y}{x}\right)$
2. If $u = x^2 + y^2 + z^2$, prove that $xu_x + yu_y + zu_z = 2u$.